HL Paper 2 Mock A 2020 – WORKED SOLUTIONS v1

Section A

1. (a) 100 students

(b) $Q_1 = \frac{n}{4} = \frac{800}{4} = 200$, $Q_3 = \frac{3n}{4} = 3 \cdot \frac{800}{4} = 600$

$a = \text{mark}(Q_1) = \text{mark}(200) = 55$

$b = \text{mark}(Q_3) = \text{mark}(600) = 75$

Hence, $a = 55$, $b = 75$

2. (a) Value after 1 year $= 3000 \times 1.046$

Value after 2 years $= (3000 \times 1.046) \times 1.046 = 3000 \times 1.046^2$

Value after n years $= 3000 \times 1.046^n$

Thus, value after 7 years $= 3000 \times 1.046^7 = \$4110.01$

(b) $5000 = 3000 \times 1.046^x \Rightarrow 1.046^x = \frac{5}{3} \Rightarrow x \ln(1.046) = \ln\left(\frac{5}{3}\right)$

$$\Rightarrow x = \frac{\ln\left(\frac{5}{3}\right)}{\ln(1.046)} = 11.3584\ldots$$

The investment will exceed \$5000 after a minimum of 12 full years

Hence, $x = 12$

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3. (a) $\dfrac{x-4}{2x^2-x-1}=\dfrac{x-4}{(2x+1)(x-1)}=\dfrac{\mathrm{A}}{2x+1}+\dfrac{\mathrm{B}}{x-1},\ \mathrm{A},\ \mathrm{B}\in\mathbb{R}$

$$\frac{x-4}{(2x+1)(x-1)}=\frac{\mathrm{A}(x-1)}{(2x+1)(x-1)}+\frac{\mathrm{B}(2x+1)}{(2x+1)(x-1)}\ \Rightarrow\ x-4=\mathrm{A}(x-1)+\mathrm{B}(2x+1)$$

Let $x=1$:

$$1-4=\mathrm{A}(1-1)+\mathrm{B}(2(1)+1)\ \Rightarrow\ 3\mathrm{B}=-3\ \Rightarrow\ \mathrm{B}=-1$$

Let $x=0$:

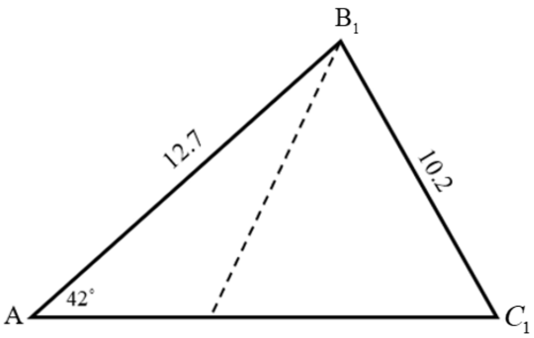
$$0-4=\mathrm{A}(0-1)-(2(0)+1)\ \Rightarrow\ -4=-\mathrm{A}-1\ \Rightarrow\ \mathrm{A}=3$$

Hence, $\dfrac{x-4}{2x^2-x-1}=\dfrac{3}{2x+1}-\dfrac{1}{x-1}$ Q.E.D.

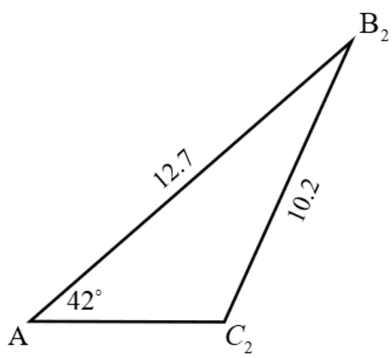
(b) $\displaystyle\int\frac{x-4}{2x^2-x-1}\mathrm{d}x=\int\left(\frac{3}{2x+1}-\frac{1}{x-1}\right)\mathrm{d}x=3\int\frac{1}{2x+1}\mathrm{d}x-\int\frac{1}{x-1}\mathrm{d}x$

$$\Rightarrow\ \int\frac{x-4}{2x^2-x-1}\mathrm{d}x=\frac{3}{2}\ln|2x+1|-\ln|x-1|+C$$

4.



OR



$\dfrac{\sin 42^\circ}{10.2}=\dfrac{\sin C_1}{12.7}\ \Rightarrow\ C_1=\sin^{-1}\left(\dfrac{12.7\sin 42^\circ}{10.2}\right)$

$C_1=56.442^\circ\ \Rightarrow\ B_1=180^\circ-(56.422^\circ+42^\circ)$

$B_1=81.578^\circ\ \Rightarrow\ \dfrac{\sin 81.578^\circ}{AC_1}=\dfrac{\sin 42^\circ}{10.2}$

$AC_1=\dfrac{10.2\sin 81.578^\circ}{\sin 42^\circ}=15.079\,\text{cm}$

$C_2=180^\circ-56.422^\circ=123.578^\circ$

$B_2=180^\circ-(123.578^\circ+42^\circ)=14.422^\circ$

$\dfrac{\sin 14.222^\circ}{AC_2}=\dfrac{\sin 42^\circ}{10.2}$

$AC_2=\dfrac{10.2\sin 14.422^\circ}{\sin 42^\circ}=3.7966\,\text{cm}$

Hence, the two possible lengths of AC are 15.1 cm and 3.80 cm

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5. (a) (i) binomial distribution: $n=500,\ \ p=\frac{3}{5}=0.6$

$$\mathrm{E}(X)=np=500(0.6)=300$$

(ii) standard deviation $=\sqrt{\mathrm{Var}(X)}=\sqrt{np(1-p)}=\sqrt{500(0.6)(0.4)}=10.9545...\approx 11.0$

(b) $\mathrm{P}(300-10.9545...<X<300+10.9545...)=\mathrm{P}(289.046...<X<310.954...)\approx 0.662$

6. Let X be the random variable representing time (in minutes) it takes for a student to travel to school

$$\mathrm{P}(X<5)=0.04 \ \Rightarrow\ Z\approx -1.75069...$$

$$\mathrm{P}(X<25)=0.7 \ \Rightarrow\ Z\approx 0.524401...$$

Using formula for standardized normal variable $Z=\frac{x-\mu}{\sigma}$

$$-1.75069...=\frac{5-\mu}{\sigma} \ \Rightarrow\ \mu-1.75069\sigma=5$$

$$0.524401...=\frac{25-\mu}{\sigma} \ \Rightarrow\ \mu+0.524401\sigma=25$$

Solving system of linear equations: $\mu\approx 20.4\,\mathrm{min},\ \ \sigma\approx 8.79\,\mathrm{min}$

7. (a) Using GDC, $\frac{1+x}{(1-4x)^3}$ can be expressed as the sum of two fractions:

$$\frac{1+x}{(1-4x)^3}=\frac{5}{4(1-4x)^3}-\frac{1}{4(1-4x)^2}=\frac{5}{4}(1-4x)^{-3}-\frac{1}{4}(1-4x)^{-2}$$

By applying the binomial expansion theorem up to and including the x^3 term:

$$\frac{5}{4}(1-4x)^{-3}\approx\frac{5}{4}\left(1+12x+96x^2+640x^3\right)$$

$$\frac{1}{4}(1-4x)^{-2}\approx\frac{1}{4}\left(1+8x+48x^2+256x^3\right)$$

$$\Rightarrow\ \frac{5}{4}(1-4x)^{-3}-\frac{1}{4}(1-4x)^{-2}\approx\frac{5}{4}\left(1+12x+96x^2+640x^3\right)-\frac{1}{4}\left(1+8x+48x^2+256x^3\right)$$

$$=1+13x+108x^2+736x^3$$

Hence, $\frac{1+x}{(1-4x)^3}\approx 1+13x+108x^2+736x^3$

(b) For the expansion to be valid, i.e. for it to converge:

$$|4x|<1 \ \Rightarrow\ -1<4x<1 \ \Rightarrow\ -\frac{1}{4}<x<\frac{1}{4}$$

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8. $$v(t)=\int a(t)\,\mathrm{d}t=\int\left(\frac{3}{t}+2t\sin 2t\right)\mathrm{d}t=3\int\frac{1}{t}\,\mathrm{d}t+2\int t\sin 2t\,\mathrm{d}t$$

To find $\int t\sin 2t\,\mathrm{d}t$, apply integration by parts:

$$\int u\,\mathrm{d}v=uv-\int v\,\mathrm{d}u$$

Let $u=t \Rightarrow \mathrm{d}u=\mathrm{d}t$, and let $\mathrm{d}v=\sin 2t\,\mathrm{d}t \Rightarrow v=\int\sin 2t\,\mathrm{d}t=-\frac{1}{2}\cos 2t$

$$\int t\sin 2t\,\mathrm{d}t=-\frac{t}{2}\cos 2t-\int-\frac{1}{2}\cos 2t\,\mathrm{d}t$$

$$=-\frac{t}{2}\cos 2t+\frac{1}{4}\sin 2t+C$$

Substitute back into our original equation, along with $3\int\frac{1}{t}\,\mathrm{d}t=3\ln t$:

$$\Rightarrow\ v(t)=3\ln t-t\cos 2t+\frac{1}{2}\sin 2t+C$$

At $t=1$, the particle is at rest, i.e. $v(1)=0$, so

$$v(1)=3\ln 1-\cos 2(1)+\frac{1}{2}\sin 2(1)+C=0$$

$$\Rightarrow\ C=\cos 2-\frac{1}{2}\sin 2=-0.8708\ldots$$

At $t=6$:

$$v(6)\approx 3\ln 6-6\cos 2(6)+\frac{1}{2}\sin 2(6)-0.8708=-0.8269\ldots$$

Hence, $v(6)\approx-0.827\,\mathrm{m\,s^{-1}}$

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9. $px^2+qx+q=0 \Rightarrow x^2+\frac{q}{p}x+\frac{q}{p}=0$

α and β are roots of the equation, so

$$x^2+\frac{q}{p}x+\frac{q}{p}=(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$$

$$\Rightarrow \alpha+\beta=-\frac{q}{p},\ \alpha\beta=\frac{q}{p}$$

The equation with roots $\frac{1}{\alpha+1}$ and $\frac{1}{\beta+1}$ can be expressed as

$$\left(x-\frac{1}{\alpha+1}\right)\left(x-\frac{1}{\beta+1}\right)=0 \Rightarrow x^2-\left(\frac{1}{\alpha+1}+\frac{1}{\beta+1}\right)x+\left(\frac{1}{\alpha+1}\right)\left(\frac{1}{\beta+1}\right)=0$$

Focusing on the constant term:

$$\frac{1}{\alpha+1}\cdot\frac{1}{\beta+1}=\frac{1}{(\alpha+1)(\beta+1)}=\frac{1}{\alpha\beta+\alpha+\beta+1}=\frac{1}{\frac{q}{p}-\frac{q}{p}+1}=1$$

For the x term:

$$\frac{1}{\alpha+1}+\frac{1}{\beta+1}=\frac{\beta+1}{(\alpha+1)(\beta+1)}+\frac{\alpha+1}{(\alpha+1)(\beta+1)}=\alpha+\beta+2=-\frac{q}{p}+2$$

Rewriting the new equation:

$$x^2-\left(-\frac{q}{p}+2\right)x+1=0$$

Since we want integer coefficients, and $p,\ q\in\mathbb{Z}$, multiply through by p:

$$px^2+(q-2p)x+p=0 \qquad \text{Q.E.D.}$$

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Section B

10. (a) Input data into GDC to determine the linear regression equation L_1 :

$y = 10.7x + 121$ (values accurate to 3 significant figures)

(b) (i) gradient of regression equation is additional cost per box, i.e. **unit cost**

(ii) y-intercept of regression equation is the **fixed costs**, i.e. cost when zero boxes are produced

(c) $y = 10.6555(60) + 120.794 = 760.124$

Hence, cost of 60 boxes is approximately $760

(d) $19.99x > y = 10.6555x + 120.794$

$\Rightarrow\ 9.3345x > 120.794\ \Rightarrow\ x > 12.9405\ldots$

Hence, the factory must produce at least 13 boxes per day to make a profit

(e) This would be extrapolation, which is not appropriate

(f) $\text{L}_2 :\ x = 0.0844y - 7.88$

(g) $x = 0.08837(550) - 7.88307 = 40.72043$

Hence, approximately 41 boxes are produced when total production cost is $550

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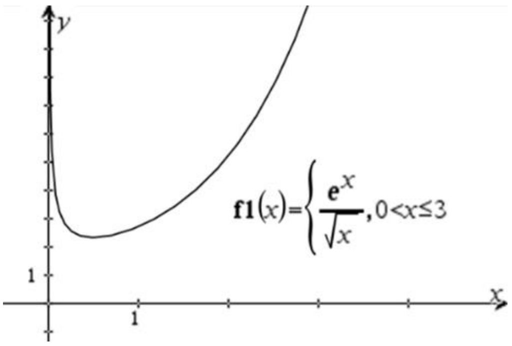
11. (a) (i)



(ii) $h(x)=\dfrac{e^x}{\sqrt{x}}=\dfrac{e^x}{x^{\frac{1}{2}}} \Rightarrow h^{-1}(x)=\dfrac{x^{\frac{1}{2}}e^x-\frac{1}{2}x^{-\frac{1}{2}}e^x}{\left(x^{\frac{1}{2}}\right)^2}=\dfrac{e^x\left(\sqrt{x}-\dfrac{1}{2\sqrt{x}}\right)}{x}=\dfrac{e^x\left(\dfrac{2x-1}{2\sqrt{x}}\right)}{x}=e^x\left(\dfrac{2x-1}{2x\sqrt{x}}\right)$

(iii) gradient of normal to curve is $-\dfrac{2x\sqrt{x}}{e^x(2x-1)}=\dfrac{2x\sqrt{x}}{e^x(1-2x)}$

(b) (i) gradient of (PQ) is $\dfrac{y-0}{x-1}=\dfrac{\dfrac{e^x}{\sqrt{x}}-0}{x-1}=\dfrac{e^x}{\sqrt{x}}\cdot\dfrac{1}{x-1}=\dfrac{e^x}{\sqrt{x}(x-1)}$

(ii) Equating the two expressions for gradient of normal to the curve gives

$\dfrac{e^x}{\sqrt{x}(x-1)}=\dfrac{2x\sqrt{x}}{e^x(1-2x)} \Rightarrow x\approx 0.545428...$ this is the x-coordinate of P

y-coordinate of P is $h(0.545428...)=\dfrac{e^x}{\sqrt{0.545428...}}\approx 2.33619...$

minimum distance from Q to graph of h is length of PQ

hence, minimum distance $=\sqrt{(0.545428...-1)^2+(2.33619...-0)^2}\approx 2.380001...$

minimum distance from Q to graph of h is approximately 2.38

(c) $g(x)=\dfrac{e^x}{c\sqrt{x}}=\dfrac{1}{c}\cdot h(x) \Rightarrow g'(x)=\dfrac{1}{c}\cdot h'(x)=\dfrac{e^x}{c}\left(\dfrac{2x-1}{2x\sqrt{x}}\right)$

The point on the graph of g nearest to point R on the x-axis is the point on g that has a horizontal tangent (parallel to x-axis); i.e. $g'(x)=0$. This point on g and point R have the same x-coordinate.

$g'(x)=\dfrac{e^x}{c}\left(\dfrac{2x-1}{2x\sqrt{x}}\right)=0 \Rightarrow 2x-1=0 \Rightarrow x=\dfrac{1}{2}$; Thus, point R is located at $\left(\dfrac{1}{2},0\right)$

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12. (a) k will always be positive. The temperature of the egg is decreasing which means that $\frac{dT}{dt} < 0$;

and since $T - A > 0$ then it must follow that $-k < 0$; hence, $k > 0$

(b) (i) $\frac{dT}{dt} = -k(T-18) \Rightarrow \frac{1}{T-18}dT = -k\,dt \Rightarrow \int \frac{1}{T-18}dT = -k\int dt$

$\ln(T-18) = -kt + C_1 \Rightarrow e^{\ln(T-18)} = e^{-kt+C_1} \Rightarrow T-18 = e^{-kt}e^{C_1}$ let $C = e^{C_1}$

$T(t) = Ce^{-kt} + 18$; given $T(0) = 98$ then $T(0) = Ce^0 + 18 = 98 \Rightarrow C = 80$

Thus, $T(t) = 80e^{-kt} + 18$ ***Q.E.D.***

(ii) Given $T(5) = 38$, find k: $38 = 80e^{-k(5)} + 18 \Rightarrow 80e^{-5k} = 20 \Rightarrow e^{-5k} = 0.25$

$\ln(e^{-5k}) = \ln(0.25) \Rightarrow -5k = \ln(0.25) \Rightarrow k = \frac{\ln(0.25)}{-5} \approx 0.277259...$

$20 = 80e^{-0.277t} + 18 \Rightarrow e^{-0.277t} = 0.025 \Rightarrow \ln(e^{-0.277t}) = \ln(0.025)$

$-0.277t = \ln(0.025) \Rightarrow t = \frac{\ln(0.025)}{-0.277259...} \approx 13.3048...$

Thus, it takes approximately 13.3 minutes for the egg to cool to 20 °C.

(c) (i) substituting, gives $\frac{dT}{dt} = -0.25(T - 18e^{-0.2t})$; then $\frac{dT}{dt} = -0.25T + 4.5e^{-0.2t}$ ***Q.E.D.***

(ii) this is a first order linear differential equation: $\frac{dT}{dt} + 0.25T = 4.5e^{-0.2t}$

integrating factor is $e^{\int 0.25\,dt} = e^{0.25t}$; multiply both sides of diff eqn by integrating factor – and applying product rule for differentiation 'backwards' on left side, gives

$e^{0.25t}\left(\frac{dT}{dt} + 0.25T\right) = e^{0.25t}(4.5e^{-0.2t}) \Rightarrow \frac{d}{dt}(e^{0.25t}T) = 4.5e^{0.05t}$

integrate both sides w.r.t. t: $\int\left[\frac{d}{dt}(e^{0.25t}T)\right]dt = 4.5\int e^{0.05t}dt \Rightarrow e^{0.25t}T = 4.5(20e^{0.05t}) + C$

$\frac{e^{0.25t}T}{e^{0.25t}} = \frac{90e^{0.05t} + C}{e^{0.25t}} \Rightarrow T(t) = 90e^{-0.2t} + Ce^{-0.25t}$

$T(0) = 98$: $98 = 90e^0 + Ce^0 \Rightarrow C = 8$

Thus, $T(t) = 90e^{-0.2t} + 8e^{-0.25t}$

(iii) $90e^{-0.2t} + 8e^{-0.25t} = 20 \Rightarrow t \approx 7.81242...$

Thus, it takes approximately 7.81 minutes for the egg to cool to 20 °C.